Lecture 8

Last week:

Relationship between total positivity and conjugate classes

Thm. 1. Let $w_1, w_2 \in W$. Then $G_{w_1,w_2,>0} \cap G^{reg.ss} \neq \phi$ 2. $G_{w_1,w_2,>0} \subseteq G^{reg.ss}$ iff $supp(w_1) = supp(w_2) = I$ 3. $G^{uni} \cap G_{\geq 0} = \bigsqcup_{w_1,w_2 \in W,\ supp(w_1)\cap supp(w_2)=\phi} U^+_{w_1,>0} U^-_{w_2,>0}$

We have prove 1) and 2) based on prop (*).

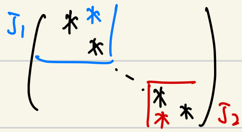
Now we prove 3):

proof: (i) If $supp(w_1) \cap supp(w_2) = \phi$, then $U^+_{w_1,>0} U^-_{w_2,>0} = G^{uni} \cap G_{w_1,w_2,>0}$

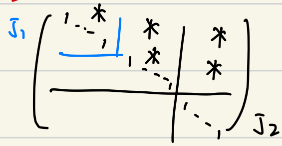
(ii) If $supp(w_1) \cap supp(w_2) \neq \phi$, then $U^+_{w_1,>0} U^-_{w_2,>0} \cap G^{uni} = \phi$

(i) Let $J_1 = supp(w_1)$, $J_2 = supp(w_2)$ s.t. $J_1 \cap J_2 = \phi \Rightarrow G_{w_1,w_2,>0} = U^+_{w_1,>0} T_{>0} U^-_{w_2,>0}$

Example: $G = GL_n$



The unipotent radical of P_{J_2}



In particular, $U^+_{w_1,>0} \subseteq U^+ \cap L_{J_1} \subseteq U_{P^+_{J_2}}$

Let w_{J_2} be the longest element of W_{J_2}. $\dot{w}_{J_2}$ a representative of $w_{J_2} \Rightarrow \dot{w}_{J_2} \in L_{J_2}$

As conjugate by $\dot{w}_{J_2}$ stabilize $U_{P^+_{J_2}}$ and sends $U^- \cap L_{J_2}$ to $U^+ \cap L_{J_2} \Rightarrow$ let $g = u_1 t u_2 \in G_{w_1,w_2,>0}$, $u_1 \in U^+_{w_1,>0}$, $t \in T_{>0}$, $u_2 \in U^-_{w_2,>0}$. conjugate by $\dot{w}_{J_2}$, we obtain $\dot{w}_{J_2} g \dot{w}_{J_2}^{-1} =$

$(\dot{w}_{J_2} u_1 \dot{w}_{J_2}^{-1})(\dot{w}_{J_2} t \dot{w}_{J_2}^{-1})(\dot{w}_{J_2} u_2 \dot{w}_{J_2}^{-1}) \Rightarrow \dot{w}_{J_2} g \dot{w}_{J_2}^{-1}$ is unipotent iff $\dot{w}_{J_2} t \dot{w}_{J_2}^{-1} = 1$ iff

$$\underset{U_{P_{J_2}^+}}{\uparrow} \qquad \underset{T}{\uparrow} \qquad \underset{U^+ \cap L_J}{\uparrow}$$

$t = 1 \Rightarrow G_{w_1, w_2, >0} \cap G^{uni} = U^+_{w_1, >0} U^-_{w_2, >0}$

(ii) If $J_1 \cap J_2 \neq \phi$, let $K = J_1 \cap J_2$, we use the idea from last week

Let $g = u_1 t u_2 \in G_{w_1, w_2, >0}$. Then similar to the proof of part 1, we have $u^{-1} g u \in$ $(L_{J_1} \cap U_{P_K^-}) u' t' (U_{P_{J_K}^+} \cap L_J)$ $(*)$. Here by prop$(*)$, $d_i(t') > 1$ $\forall i \in K$.

In particular, $t' \neq 1$

Now we may conjugate $(*)$ by a suitable element in the Weyl group to an element in $U^+ t''$, where t'' is conjugate to t'. As $t' \neq 1 \Rightarrow t'' \neq 1 \Rightarrow$ no element in $U^+ t''$ is unipotent $\Rightarrow G_{w_1, w_2, >0} \cap G^{uni} = \phi$. $\boxtimes$

Remarks: 1. Let G be a reductive group over $\mathbb{C}$. Then $\dim_{\mathbb{C}} G^{uni} = \#$roots of $G = 2L(w_0)$ $= \dim_{\mathbb{C}} G - \operatorname{rank} G$. where w_0 is the longest element of W. If $G = Gl_n$, $\dim_{\mathbb{C}} G^{uni} = n^2 - n$.

2. $\dim_{\mathbb{R}} G_{\geq 0} = \dim_{\mathbb{C}} G^{(*)}$ but $\dim_{\mathbb{R}} (G_{\geq 0} \cap G^{uni}) = \max_{w_1, w_2 \in W, \, supp(w_1) = supp(w_2) = \phi} \dim_{\mathbb{R}} U^+_{w_1, >0} U^-_{w_2, >0}$ $= \max_{supp(w_1) \cap supp(w_2) = \phi} L(w_1) + L(w_2) = L(w_0) \Rightarrow \dim_{\mathbb{R}} (G_{\geq 0} \cap G^{uni}) = \frac{1}{2} \dim_{\mathbb{C}} (G^{uni})$

3. For $(*)$ $\quad G_{\geq 0} = \bigsqcup_{w_1, w_2 \in W} G_{w_1, w_2, >0} = \bigsqcup U^+_{w_1, >0} T_{>0} U^-_{w_2, >0} \Rightarrow \dim_{\mathbb{R}} (G_{w_1, w_2, >0}) = l(w_1) + \dim(T_{>0})$ $+ l(w_2) = l(w_1) + \operatorname{rank} G + l(w_2) \Rightarrow \dim G_{\geq 0} = \max_{w_1, w_2 \in W} l(w_1) + \operatorname{rank} G + l(w_2) = l(w_0) + \operatorname{rank} G =$ $\dim G$.

4. $G^{reg, ss} \subseteq G$ open dense. $G_{>0} \subseteq G^{reg, ss} \cap G_{\geq 0} \subseteq G_{\geq 0}$ open dense $\Rightarrow \dim_{\mathbb{R}} (G^{reg, ss} \cap G_{\geq 0})$

$$= \dim_{\mathbb{C}} (G^{reg,ss}) \qquad st: G \longrightarrow G^{ss} / G\text{-conjugation}$$

generic one are reg.ss whose fiber is trivial

fibers corr. to unipotent conjugate classes

Open problem: $G^{uni} \cap G_{\geq 0} = \bigsqcup_{w_1, w_2 \in W \; supp(w_1) \cap supp(w_2) = \phi} U^+_{w_1, \geq 0} U^-_{w_2, \geq 0}$

For each w_1, w_2 with $supp(w_1) \cap supp(w_2) = \phi$. which unipotent conjugate class intersects $U^+_{w_1,\geq 0} U^-_{w_2, \geq 0}$?

Related question: Let $J_1, J_2 \subseteq I$ with $J_1 \cap J_2 = \phi$. Consider $(U^+ \cap L_{J_1}^{reg,uni})(U^- \cap L_{J_2}^{reg,uni})$ instead of $U^+_{w_1, \geq 0} U^-_{w_2, \geq 0}$.

Example: $G = Gl_n$ $\quad supp(w_i) = J_i$. unipotent conjugation $\longleftrightarrow$ partition of n.

$$\{(J_1, J_2) \mid J_1 \cap J_2 = \phi\} \xrightarrow{?} \{\text{partition of } n\}$$

If J_1, J_2 are not connected in the Dynkin diagram (for $G = Gl_n$, $|i-j| \geq 2$ $\forall i \in J_1, j \in J_2$.

the above map is $(J_1, J_2) \mapsto [J_1 \sqcup J_2]$

For $G = Gl_4$ $\qquad J_1 = 1, \; J_2 = 3$

$$\begin{pmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & \\ & & * & 1 \end{pmatrix} \xrightarrow{conj} \begin{pmatrix} 1 & * & & \\ & 1 & & \\ & & 1 & * \\ & & & 1 \end{pmatrix}$$

$$G_{w_1, w_2, \geq 0} \xrightarrow{conj} \text{unipotent class corr. to } 2^2$$

For $G = Gl_3$ $\qquad J_1 = 1, \; J_2 = 2$

$$\begin{pmatrix} 1 & * & \\ & 1 & \\ & * & 1 \end{pmatrix} \xrightarrow{\dot{w}_{J_2}} \begin{pmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{pmatrix}$$

$G_{w_1,w_2,>0} \longrightarrow$ unipotent conj corr. to $(2,1)$

$w_{J_2} = \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}$

Q: what is the general pattern?

Guess: $\forall w_1, w_2 \in W$ with $\mathrm{supp}(w_1) \cap \mathrm{supp}(w_2) = \phi$, $\exists !$ unipotent conjugacy class C s.t. $U^+_{w_1,>0} U^-_{w_2,>0} \subset C$. Moreover, C only depends on J_1 and J_2 and C is the regular unipotent conjugacy class of some Levi subgroup L_J.

In particular, we have a map $\{(J_1, J_2) \mid J_1, J_2 \subset I,\ J_1 \cap J_2 = \phi\} \longrightarrow \{J \subseteq I\}/$ conjugation.

More about the Steinberg maps:

$St: \quad G \longrightarrow G//G = T/W \qquad g \mapsto$ conj class of g_s

Two extreme cases 1) $g_s = 1 \Rightarrow$ the fiber is G^{uni} $\exists !$ regular conj class in G^{uni}, the regular unipotent one. 2) g_s is reg. ss $\Rightarrow$ the fiber is trivial as $Z_G(g_s)^{uni} = \{1\}$, $\exists !$ regular conj class in $Z_G(g_s)^{uni} = \{1\}$, $\{g_s\}$ is the regular conjugacy class of G.

In general, fix g_s s.s. element in G. $\{g' \in G \mid g'_s = g_s\} = g_s Z_G(g_s)^{uni}$

conjugacy classes of g' in $G \longleftrightarrow$ unipotent conjugacy classes of $Z_G(g_s)$

regular element $g_s u$ of $G \longleftrightarrow$ regular unipotent u

$\forall$ group H, $h \in H$, we write $H \cdot h$ for the conj class of h in $H \Rightarrow \dim G\cdot(g_s u) = \dim G \cdot g_s + \dim Z_G(g_s)\cdot u$

open problem: Jordan decompositition for TP (in H-Lusztig)

What is TP for G?

We fix $(T, B^{\pm}, x_i, y_i)$ $\longrightarrow G_{\geq 0} = \langle T_{>0}, x_i(>0), y_i(>0) \rangle$ — pinning

If we change to another pinning $(T', B'^{\pm}, x_i', y_i') \longrightarrow G'_{\geq 0}$

St: $\quad G_{\geq 0} \longrightarrow T_{>0}/W$

Conjecture: $\forall g \in G_{\geq 0}$, $\exists$ a pining on $Z_G(g_s)$ (connected reductive group) s.t. $\{g' \in G_{\geq 0} \mid g'_s = g_s\} = g_s [Z_G(g_s)^{uni}_{\geq 0}]$

Remark: 1. If $G = GL_n$, $Z_G(g_s)$ = product of GL's.

2. $\{g' \in G \mid g'_s = g_s\} = g_s Z_G(g_s)^{uni}$

Perron's Thm (used to prove prop (*))

Let A be a positive matrix (all entries are positive)

Def. The spectral radius $\rho(A) = \max\{|\lambda| ; \lambda \text{ is an eigenvalue of } A\}$

Perron's Thm: Let A be a positive matrix. Then A has a unique eigenvalue λ with $|\lambda| = \rho(A)$. Moreover, $\lambda > 0$, has a positive eigenvector and λ has multiplicity 1 key ingredient in the proof: Let $\|\cdot\|$ be a matrix norm

Gelfend's formula : $\rho(A) = \lim_{n\to\infty} (\|A^n\|)^{\frac{1}{n}}$

proof: If $\rho(A) < 1 \Rightarrow \lim_n A^n = 0$, $\rho(A) > 1 \Rightarrow \lim \|A^n\| \to \infty$ (use Jordan form)

Consider $A'_\pm = \frac{1}{\rho(A)\pm\varepsilon} A$, ε is a small positive number $\Rightarrow \rho(A'_\pm) = \frac{\rho(A)}{\rho(A)\pm\varepsilon} \Rightarrow \rho(A'_+) < 1$, $\rho(A'_-) > 1$. Now $\lim_n (A'_+)^n = 0 \Rightarrow \lim_n A^n = (\rho(A)+\varepsilon)^n \lim (A'_+)^n \Rightarrow \lim \|A^n\|^{\frac{1}{n}} \le \rho_A + \varepsilon$.

OTOH, $\lim A^n = (\rho_A - \varepsilon)^n \lim_n (A'_-)^n \Rightarrow \lim \|A^n\|^{\frac{1}{n}} \geq \rho_A - \varepsilon$. As ε is arbitrary $\Rightarrow \lim \|A^n\|^{\frac{1}{n}} = \rho_A$. ☒

step1: Let λ be an eigenvalue of A with $|\lambda| = \rho(A)$. then $\exists$ a positive eigenvector for λ

Let $v \in \mathbb{C}^n$, $Av = \lambda v$. Set $|v| \in \mathbb{R}^n$ by $|v| = \{|v_1|, \cdots, |v_n|\} \Rightarrow (A|v|)_i = \sum A_{ij}|v_j| \overset{(*)}{\geq}$ $|\sum A_{ij} v_j| = |\lambda v_i| = \rho(A)|v_i| \Rightarrow A|v| \geq \rho(A)|v|$ for each factor

If $A|v| \neq \rho(A)|v| \Rightarrow$ for certain factor, LHS > RHS Then apply A to both sides $\Rightarrow$ LHS has all factors > RHS (as all entries of $A > 0$) $A^2|v| > \rho(A)A|v|$ strictly for each entry $\Rightarrow \exists\, \varepsilon > 0$ s.t. $A^2|v| \geq (1+\varepsilon)\rho(A)A|v|$. Apply A again, $A^3|v| \geq (1+\varepsilon)\rho_A A^2|v| \geq ((1+\varepsilon)\rho(A))^2 A|v| \cdots$ $A^{n+1}|v| \geq ((1+\varepsilon)\rho(A))^n A|v|$. Apply the Gelfand formula $\rho(A) = \lim \|A^{n+1}\|^{\frac{1}{n+1}} \geq (1+\varepsilon)\rho(A)$ contradiction $\Rightarrow A|v| = \rho(A)|v| \Rightarrow \rho(A)$ is an eigenvalue, $|v|$ a positive eigenvector.

Also, (*) is an equality $\Rightarrow \forall i$, $\sum A_{ij}|v_j| = |\sum A_{ij} v_j| \Rightarrow \exists\, c \in \mathbb{C}$ s.t. $v_j = c|v_j|$ $\forall j$.

In other words $v = c|v| \Rightarrow v$ is also an eigenvector of A with eigenvalue $\rho(A) \Rightarrow \lambda = \rho(A)$.

step2: $\rho(A)$ has multiplicity 1

If v' is another eigenvector As all the entries of A are real, $A(\mathrm{Re}\,v')=\rho(A)(\mathrm{Re}\,v') \Rightarrow$

WLOG. assume v' is real.

Now $Av=\rho(A)v$, $Av'=\rho(A)v' \Rightarrow \exists c$ s.t. $v-cv'$ has all factors ≥ 0, one factor $=0$

$\Rightarrow \underset{\substack{\uparrow \\ \text{all the factors} > 0 \text{ unless } c-cv'=0}}{A(v-cv')} = \rho(A)(v-cv') \leftarrow$ at least one factor $=0$ $\Rightarrow v=cv' \Rightarrow$ the geometric

multiplicity $=1$.

For the algebraic multiplicity, we will try to block diagonalize A. Let w be a positive vector with $A^T w=\rho(A)w$, we choose a basis $\underbrace{\text{for } \{z\in\mathbb{C}^n : z\perp w\}}_{}$ $(*)$ Set $X=[v \text{ basis}(*)]$

$$X^{-1}AX=\begin{array}{c} 1 \qquad\quad \\ \left[\begin{array}{c|c} \rho(A) & \\ \hline & * \end{array}\right] \\ \qquad\quad n-1 \end{array}$$

block diagonal. As the geometric multiplicity of $\rho(A)$ is 1

$\Rightarrow \rho(A)$ is not an eigenvalue of $* \Rightarrow$ algebraic multiplicity of $\rho(A)=1$. $\boxtimes$